Partially ordered sets

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Posets

Definition 1

A partially ordered set (or poset, for short) is a set P together with a binary relation $\leq$ which satisfies the following three axioms:

1. $\forall x \in P : x \leq x$,
2. $\forall x, \; y \in P : (x \leq y \wedge y \leq x) \implies x = y$, *and*
3. $\forall x, \; y, \; z \in P : (x \leq y \wedge y \leq z) \implies x \leq z$.

By abuse of notation, we shall often refer to P as a poset, instead of $(P, \; \leq)$ if there's no confusion. We may also use $\leq_P$ at times when there's a possibility of confusion. We say that elements x and y of P are comparable if either $x \leq y$ or $y \leq x$. The term *partially* refers to the fact that there may be elements in the poset that are not comparable.

More notations

We also define the following three notations:

1. $x \geq y$ iff $y \leq x$,
2. $x < y$ iff $x \leq y$ and $x \neq y$, and
3. $x > y$ iff $y < x$.

We shall also concatenate things by writing $x \leq z \leq y$ to mean $x \leq z$ and $z \leq y$. We can extend this by concatenating more than three elements as well as using different operations such as $x \leq y < z \leq w$.

We shall also frequently use the following notation:
Let $\mathbb{N}$ denote the set of positive integers.
For $n \in \mathbb{N}$, define $[n] := \{k \in \mathbb{N} : k \leq n\}$.
That is, $[n]$ is the set of positive integers up to (and including) n.

Examples of posets

Here are some examples of posets. Let n be any positive integer.

1. $[n]$ with the usual ordering of integers is a poset. Moreover, any two elements are comparable.
2. Let $2^{[n]}$ denote all the subsets of $[n]$.
   We can define an ordering on $2^{[n]}$ as: $A \leq B$ if $A \subset B$. As a poset, we shall denote this by B_n.
3. Let S denote all the positive integer divisors of n.
   Define an ordering on S as: $a \leq b$ if $a | b$. As a poset, we shall denote this by D_n.
4. Let P denote the set of (set) partitions of $[n]$.
   Define an ordering on P as: $\pi \leq \sigma$ if every block of π is contained in a block of σ.
   As a poset, we shall denote this by Π_n.
   As an example, let $n = 5$. Take $\pi = [1][234][5]$ and $\sigma = [1][2345]$. Then, we have it that $\pi \leq \sigma$.
5. In general, any collection of sets can be ordered by inclusion to form a poset.

Isomorphism

Let P and Q be two posets.
An isomorphism is a map $\varphi : P \to Q$ such that φ is a bijection and

$$x \leq_P y \iff \varphi(x) \leq_Q \varphi(y) \text{ for every } x \text{ and } y \text{ in } P.$$

Two posets P and Q are said to be isomorphic if there exists an isomorphism from P to Q. We denote this by writing $P \cong Q$.

What this really means is that P and Q are identical in terms of their structure as a poset and the elements of P could simply be relabeled to give Q.

Subposets

Definition 2 (Weak subposet)

By a weak subposet of P, we mean a subset Q of P together with a partial ordering of Q such that $x \leq_Q y \implies x \leq_Q y$ for all x and y in Q.

If Q is a weak subposet of P and $Q = P$ as sets, then P is called a *refinement* of Q.

Definition 3 (Induced subposet)

By an induced subposet of P, we mean a subset Q of P together with a partial ordering of Q such that $x \leq_Q y \iff x \leq_Q y$ for all x and y in Q.

Unless otherwise mentioned, by a subposet of P, we shall always mean an induced subposet.

If $|P| < \infty$, then there exist exactly $2^{|P|}$ induced subposets of P.

Intervals

Definition 4

A special subposet of P is the (closed) interval $[x, y] = \{z \in P : x \leq z \leq y\}$ defined whenever $x \leq y$.

By definition, it should be clear that $\emptyset$ is *not* an interval.
Also, note that $[x, x] = \{x\}$.

Definition 5 (Locally finite poset)

If every interval of P is finite, then P is called a locally finite poset.

Examples of locally finite posets are: B_n, $\mathbb{N}$, $\mathbb{Z}$.
Examples of non-locally finite posets are: $2^{\mathbb{N}}$, $\mathbb{R}$, $\mathbb{Q}$.
($2^{\mathbb{N}}$ denotes the power set of $\mathbb{N}$ which is a poset when ordered by inclusion.)
($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ have their usual ordering.)

Finite posets

Definition 6

A poset $(P, \leq)$ is said to be finite if P is finite.

Every finite poset is locally finite but the converse is not true as we saw earlier in the case of $\mathbb{Z}$.

Convexity and covering

Definition 7 (Convex subposets)

We define a subposet Q of P to be convex if $y \in Q$ whenever $x < y < z$ and $x, z \in Q$.

Thus, an interval is always convex.

Definition 8 (Cover)

If $x, y \in P$, then we say that y covers x if $x < y$ and $\nexists z \in P$ such that $x < z < y$.

The above is equivalent to saying that $x < y$ and $[x, y] = \{x, y\}$.
A locally finite poset P is completely determined by its cover relations.

Hasse diagrams

The Hasse diagram of a finite poset P is the graph whose vertices are the elements of P, whose edges are cover relations, and such that if $x < y$, then y is drawn "above" x.

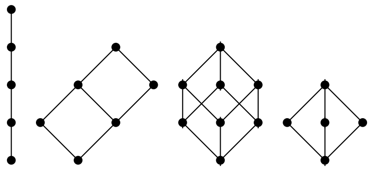


Figure: Hasse Diagrams of $[5]$, D_{12}, B_3, and Π_3

Note that given the same poset, one may make different *looking* Hasse diagrams. If two posets have the same Hasse diagram, then they are clearly isomorphic.

$\hat{0}$ and $\hat{1}$

We say that P has a $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$.
Similarly, P has a $\hat{1}$ is there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$.
We denote by $\hat{P}$ the poset obtained by adjoining a $\hat{0}$ and a $\hat{1}$ to P. This is regardless of whether or not P had a $\hat{0}$ or a $\hat{1}$ to begin with.
Note that $\hat{0}$ and $\hat{1}$ *have* to comparable with every element, by definition.

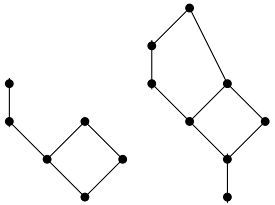


Figure: P and $\hat{P}$

Extremal elements

Definition 9

We say that $x \in P$ is a minimal element if $y \leq x \implies y = x$ for all $y \in P$.

Definition 10

We say that $x \in P$ is a maximal element if $y \geq x \implies y = x$ for all $y \in P$.

Note that a poset may not have a minimal or a maximal element to begin with.
Example - $\mathbb{N}$
Even if a minimal (or maximal) element exists, it need not be unique. Example - $\{2, 3\}$ regarded as a subposet of D_6. All the elements are minimal as well as maximal.
The above example also illustrates that a minimal (maximal) element need not necessarily be $\hat{0}$ ($\hat{1}$). This sort of behaviour is precisely due to the fact that two elements may not be comparable.

Chains

Definition 11

A chain (or totally ordered set) is a poset in which any two elements are comparable.

Definition 12

A subset C of P is called a chain if C is a chain when regarded as a subposet of P.

Definition 13

A chain C of P is called saturated (or unrefinable) if there does not exist $z \in P \setminus C$ such that $x < z < y$ for some $x, y \in C$ and $C \cup \{z\}$ is still a chain.

Definition 14

A chain C of P is called maximal if there does not exist $z \in P \setminus C$ such that $C \cup \{z\}$ is still a chain.

Examples

Consider $P = D_{30}$ and the following subsets of P :

1. $C_1 = \{1, 15, 30\}$. C_1 is a chain but not saturated as $1 < 5 < 15$ and $C_1 \cup \{5\}$ is still a chain. For similar reasons, it is not maximal either.
2. $C_2 = \{1, 5, 15\}$. C_2 is a chain. It is saturated as well. However, it is not maximal.
3. $C_3 = \{1, 5, 15, 30\}$ is a maximal (and saturated) chain.
4. $C_4 = P$ is not a chain. Note that C_4 is an interval. Thus, intervals need not be chains.

In a locally finite poset, a chain $x_0 < x_1 < \cdots < x_n$ is saturated if and only if x_i covers x_{i-1} for all $i \in [n]$.

Lengths

Definition 15

The length of a finite chain C is denoted by $l(C)$ and is defined as $l(C) := |C| - 1$.

Definition 16

The length (or rank) of a finite poset is $l(P) := \max\{l(C) : C \text{ is a chain of} P\}$.

The length of an interval $[x, y]$ is denoted by $l(x, y)$.

Definition 17

If every maximal chain of P has the length $n \in \mathbb{N} \cup \{0\}$, then we say that P is graded of rank n.

Before proving a result about graded posets, let us see the notion of something known as a *rank function.*

Rank function

Definition 18

A rank function of a poset P is a function $\rho : P \to \mathbb{N} \cup \{0\}$ having the following properties:

1. *if x is minimal, then $\rho(x) = 0$, and*
2. *if y covers x, then $\rho(y) = \rho(x) + 1$.*

Note that saying "*a* rank function" instead of "*the* rank function" has a subtlety. Given an arbitrary poset P, it is **not** necessary that is has a rank function. For example, $\mathbb{Z}$ has no rank function. Also, given a poset P, it *may* have more than one rank functions as well. As an example, the set of nonnegative real numbers has infinitely many rank functions!

Even a finite poset need not have a rank function. Example- $\{2, 6, 15, 30\}$ regarded as a subposet of D_{30}.

Some rank theorems

Theorem 1

Every graded poset possesses a unique rank function.

It is important to observe that even if the poset is not finite, it could still be graded. For example, $(\mathbb{N}, =)$ is graded of rank 0.
Before we prove Theorem 1, let us see another theorem.

Theorem 2

If $x \leq y$, then $l(x, y) = \rho(y) - \rho(x)$.

Given an element x of a graded poset, the existence and uniqueness of a rank function lets us talk about the rank of x. We define rank of x to be $\rho(x)$, where ρ is the unique rank function.

Lemma 1

Every finite chain possesses a unique rank function.

Proof.

Assume $C = \{x_0, x_1, \ldots, x_n\}$ is a finite chain of length n such that $x_0 < x_1 < \cdots x_n$. Then, x_0 is a minimal element of C, and for all $i \in [n]$, we have it that x_i covers x_{i-1}. Define $\rho : C \to \mathbb{N} \cup \{0\}$ by defining $\rho(x_i) = i$. Then, ρ satisfies the properties of a rank function. This shows the existence of a rank function.

Suppose ρ' were another rank function of C different from ρ. It is forced that $\rho'(x_0) = 0$. Thus, for some $i \in [n]$, $\rho(x_i) \neq \rho'(x_i)$.
If $\rho(x_i) < \rho(x_i)'$, then $\rho'(x_0) = \rho'(x_1) - 1 = \cdots = \rho'(x_i) - i > i - i = 0$, a contradiction.
Similarly, if $\rho(x_i) < \rho(x_i)'$, we get that $\rho'(x_0) < 0$, a contradiction. □

Proof of Theorem 1

Assume P is a graded poset of rank n. Let $C = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary maximal chan of P such that $x_0 < x_1 < \cdots < x_n$. By Lemma 1, there exists a unique rank function ρ_C for C.
Let $C' = \{x'_0, x'_1, \ldots, x'_n\}$ be any other maximal chain such that $x'_0 < x'_1 < \cdots < x'_n$ and $C \cap C' \neq \emptyset$. Let $\rho_{C'}$ be the unique rank function for C' and suppose that for some $x \in C \cap C'$, $\rho_C(x) \neq \rho_{C'}(x)$.
Then, there exist $i, j \in [n] \cup \{0\}$ such that $i \neq j$ and $x = x_i = x'_j$. Without loss of generality, we can assume that $j > i$.
Then, $\{x'_0, x_1, \ldots, x'_j = x_i, \ldots, x_n\}$ is a chain of length $j + n - i > n$, which contradicts the assumption of the rank of P.
Thus, we have shown that given any chains, their rank functions agree on the common values, if any.
Since $P = \bigcup\{C \subset P : C \text{ is a maximal chain of } P\}$, we can define $\rho(x) = \rho_C(x)$ where C is any maximal chain containing x. This map is well defined by our above exercise and its uniqueness follows from the uniqueness of each ρ_C. □

Proof of Theorem 2

Assume P is a graded poset of of rank n with rank function ρ. Given $x \leq y$ in P, let $C = \{x_0, x_1, \ldots, x_n\}$ be a maximal chain of P containing x and y such that $x_0 < x_1 < \cdots < x_n$.
Then, for some $i, j \in [n] \cup \{0\}$ such that $i < j$, we have that $x_i = x$ and $x_j = y$. This forces $l(x, y) = j - i$. Else wise, we would get that $l(C) \neq n$.
But by Theorem 1, we know that $\rho(y) = j$ and $\rho(x) = i$. □

Definition 19

If P is a finite graded poset of rank n such that for each $i \in [n] \cup \{0\}$, p_i is the number of elements of P of rank i, then the rank-generating function of P is the polynomial

$$F(p,\ x) := \sum_{i=0}^{n} p_i x^i.$$

Most of the posets we saw so far were graded. Examples - $[n]$, B_n, D_n, and Π_n.

Some examples

Poset P	Rank of $x \in P$	Rank of P	$F(P,\ x)$
$[n]$	$x-1$	$n-1$	$\sum_{i=0}^{n-1} x^i$
B_n	$\lvert x \rvert$	n	$\sum_{i=0}^{n} \binom{n}{i} x^i$
D_n	number of prime divisors of x	number of prime divisors of n	$F(B_n,\ x)$, if n is square free
Π_n	$n-\lvert x \rvert$	$n-1$	$\sum_{i=0}^{n-1} S(n, n-i) x^i$

Where $S(n,\ k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$ is a Stirling number of the second kind.

Antichains and ideals

Definition 20 (Antichain)

An antichain is a subset A of a poset P such that any two distinct elements of A are not comparable.

Definition 21 (Order ideal)

An order ideal of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$.

When $|P| < \infty$, there is a one-to-one correspondence between antichains A of P and order ideals I of P.

Given an antichain A, one can construct an order ideal I as follows:

$I = \{x \in P : x \leq y \text{ for some } y \in A\}$.

Similarly, given an order ideal I, one can construct an antichain A as follows:

$A = \{x \in I : x \text{ is a maximal element of } I\}$. $(*)$

More on order ideals

The set of all order ideals of P, ordered by inclusion, forms a poset which is denoted by $J(P)$. If I and A are related as in $(*)$, then we say that A *generates* I. If $A = \{x_1, x_2, \ldots, x_k\}$, then we write $I = \langle x_1, x_2, \ldots, x_k \rangle$ for the order ideal generated by A.

The order ideal $\langle x \rangle$ is the principal order ideal generated by x, denote Λ_x.

New posets from old

We shall now see some operations on posets that let us create new posets.

Direct sum

Definition 22 (Direct sum)

If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets on disjoint sets, then the direct sum of P and Q is the poset $P + Q$ defined on $P \cup Q$ such that $x \leq y$ in $P + Q$ if either

1. $x, y \in P$ *and* $x \leq_P y$, *or*
2. $x, y \in Q$ *and* $x \leq_Q y$.

A poset that is not (isomorphic to) a disjoint union of two nonempty posets is said to be connected.

Examples -

1. [5] is connected.
2. The subposet $\{4, 6\}$ of D_{12} is not connected. $\{4, 6\} \cong \{4\} + \{6\}$.

The disjoint union of P with itself n times is denoted by nP.

An $n-$element antichain is isomorphic to $n[1]$.

Ordinal sum

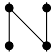
Definition 23 (Ordinal sum)

If P and Q are disjoint sets as above, then the ordinal sum of the posets P and Q, denoted by $P \oplus Q$ is the poset defined on $P \cup Q$ such that $x \leq y$ in $P \oplus Q$ if

1. $x,\ y \in P$ *and* $x \leq_P y$, *or*
2. $x,\ y \in Q$ *and* $x \leq_Q y$, *or*
3. $x \in P$ *and* $y \in Q$.

Hence, an $n-$element chain is isomorphic to $\underbrace{[1] \oplus [1] \oplus \cdots \oplus [1]}_{n \text{ times}}$.

Posets that can be built up using disjoint union and ordinal sums from the poset [1] are called series-parallel posets.



This is the only poset (up to isomorphism) with four elements that is not series-parallel.

Direct Product

Definition 24

If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets, then the direct product of P and Q is the poset $P \times Q = (P \times Q, \leq_{P\times Q})$ such that $x \leq_{P\times Q} y$ if $x \leq_P x'$ and $y \leq_Q y'$.

The direct product of P with itself n times is denoted by P^n.
To draw the Hasse diagram of $P \times Q$, (when P and Q are finite) we do the following:

1. Draw the Hasse diagram of P.
2. Replace every element $x \in P$ by a copy Q_x of Q.
3. Connect corresponding elements of Q_x and Q_y if x and y are connected in the Hasse diagram of P.

It is clear from the definition that $P \times Q \cong Q \times P$. However, using the above procedure, the Hasse diagrams may *look* completely different.

A theorem on rank generating functions

Theorem 3

If P and Q are graded with rank-generating functions $F(P, x)$ and $F(Q, x)$, then $P \times Q$ is graded and $F(P \times Q, x) = F(P, x)F(Q, x)$.

Before proving this theorem, we shall first prove the following lemma:

Lemma 2

If both P and Q have finite lengths, then $l(P \times Q) = l(P) + l(Q)$.

Proof of the lemma

Proof.

Assume P has length m and Q has length n. Given any arbitrary chains $C = \{(x_0, y_0), \ldots, (x_l, y_l)\}$ of $P \times Q$ such that $(x_0, y_0) <_{P \times Q} \cdots <_{P \times Q} (x_l, y_l)$, it follows that $X = \{x_0, \ldots, x_l\}$ is a chain of P and $Y = \{y_0, y_1, \ldots, y_l\}$ is a chain of Q. Note that for each $i \in [l]$, $(x_{i-1}, y_{i-1}) <_{P \times Q} (x_i, y_i)$ implies that $x_{i-1} <_P x_i$ or $y_{i-1} <_Q y_i$. Since $l(P) = m$, we get that $x_{i-1} < x_i$ is true for at most m many elements in $[l]$ and similarly $y_{i-1} < y_i$ is true for at most n many elements. Thus, we get that $l \leq m + n$.

Now, we actually produce a chain of length $m + n$. As P has length m, there exists a chain $C_1 = \{x_0, x_1, \ldots, x_m\}$ of P such that $x_0 <_P x_1 <_P \cdots <_P x_m$. Similarly, there exists a chain $C_2 = \{y_0, y_1, \ldots y_m\}$ of Q such that $y_0 <_Q y_1 <_Q \cdots <_Q y_m$.

Then, $\mathcal{C} = \{(x_0, y_0), (x_0, y_1), \ldots (x_0, y_n), (x_1, y_n), \ldots (x_m, y_n)\}$ is a chain of $P \times Q$ of length $m + n$. □

Proof of the theorem

Assume that P and Q are graded of rank m and n, respectively. By the previous lemma, $P \times Q$ has rank $m + n$. Now we show that $P \times Q$ is indeed graded.
Let $C = \{(x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l)\}$ be an arbitrary maximal chain of $P \times Q$ such that $(x_0, y_0) <_{P \times Q} \cdots <_{P \times Q} (x_l, y_l)$. If $l < m + n$, then there exists $i \in [l]$ such that $x_{i-1} <_P x_i$ and $y_{i-1} <_Q y_i$. (Use an argument similar to that used in the proof of the previous lemma.)
But this implies that $C \cup \{(x_{i-1}, y_i)\}$ is a chain, contradicting the maximality of C.
Thus, $l(C) = m + n$. As C was arbitrary, $P \times Q$ is graded of rank $m + n$.

Now, we shall show the relation of rank-generating functions that was stated before.

Proof of the theorem

Assume that the rank generating functions of P and Q are $\sum_{i=0}^{m} p_i x^i$ and $\sum_{i=0}^{n} q_i x^i$, respectively.

Let $x \in P$ have rank k and $y \in Q$ have rank l. We show that (x, y) has rank $k + l$. To see this, consider maximal chains $X = \{x_0, x_1, \ldots, x_m\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ of P and Q, respectively such that $x \in X$ and $y \in Y$ and $x_0 <_P x_1 <_P \cdots <_P x_m$ and $y_0 <_Q y_1 <_Q < \cdots y_n$. By Theorem 1, we have it that $x = x_k$ and $y = y_l$. The chain

$$C = \{(x_0, y_0), \ldots, (x_k, y_0), \ldots, (x_k, y_l), \ldots, (x_k, y_n), \ldots, (x_m, y_n)\}$$

of $P \times Q$ such that

$$\{(x_0, y_0) <_{P\times Q} (x_k, y_0) <_{P\times Q} (x_k, y_l) <_{P\times Q} (x_k, y_n), <_{P\times Q} (x_m, y_n)\}$$

in $P \times Q$ has length $m + n$ and so it is maximal. It follows again, by Theorem 1 that (x, y) has rank $k + l$. Thus, the number of elements of $P \times Q$ of rank j is $\sum_{i=0}^{j} p_i q_{j-i}$, which is the coefficient of x^j in $F(P, x)F(Q, x)$. □

Ordinal product

Definition 25

Ordinal product If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets, then the direct product of P and Q is the poset $P \otimes Q = (P \times Q, \leq_{P \otimes Q})$ such that $x \leq_{P \otimes Q} y$ if

1. $x = x'$ *and* $y \leq y'$, *or*
2. $x < x'$.

We state the following theorem without proof:

Theorem 4

If P and Q are graded and Q has rank r, then

$$F(P \otimes Q, x) = F(p, x^{r+1})F(Q, x).$$

In general, $P \otimes Q$ and $Q \otimes P$ don't have the same rank-generating function. Thus, they are not isomorphic.

Dual of a poset

Definition 26 (Dual poset)

Let P be a poset. We denote by P^* the poset defined on the same set as that of P such that $x \leq_{P^*} y \iff y \leq_P x$.

If P and P^* are isomorphic, then P is said to be self-dual.
There are eight posets (up to isomorphism) with 4 elements that are self-dual.