# Lattices

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**Definition 1 (Upper bound)**

*If $x$ and $y$ belong to a poset $P$, then an upper bound of $x$ and $y$ is an element $z \in P$ satisfying $x \leq z$ and $y \leq z$.*

**Definition 2 (Least upper bound)**

*A least upper bound $z$ of $x$ and $y$ is an upper bound such that every upper bound $w$ of $x$ and $y$ satisfies $z \leq w$.*

Thus, if a least upper bound of $x$ and $y$ exists, then it is clearly unique due to antisymmetry of $\leq$ . The element is denoted by $x \vee y$, read as "$x$ join $y$" or "$x$ sup $y$."

Dually, one can define a greatest lower bound of $x$ and $y$.

**Definition 3**

*A lattice is a poset for which every pair of elements has a least upper bound and greatest lower bound.*

Let $L$ be a lattice and $x, y \in L$.
One can verify that the following properties hold:

1. The operations $\wedge$ and $\vee$ are associative, commutative and idempotent, that is, $x \wedge x = x \vee x = x$,
2. $x \wedge (x \vee y) = x \vee (x \wedge y) = x$, and
3. $x \wedge y = x \iff x \leq y \iff x \vee y = y$.

In fact, one could even define a lattice axiomatically in terms of a set $L$ with the operations $\wedge$ and $\vee$ satisfying the first two properties.

# Some results

All finite lattices have $\hat{0}$ and $\hat{1}$. Indeed, if $L = \{x_1, \dots, x_n\}$. Then $x_1 \wedge \cdots \wedge x_n$ and $x_1 \vee \cdots \vee x_n$ are well-defined elements of $L$ and they are $\hat{0}$ and $\hat{1}$, respectively.

If $L$ and $M$ are lattices, then so are $L^*$, $L \times M$, $L \oplus M$. However, $L + M$ will never be lattice unless $L = \emptyset$ or $M = \emptyset$. Indeed, if one takes $x \in L$ and $y \in M$, then there exists no meet of $x$ and $y$ in $L + M$.
However, one can verify that $\widehat{L + M}$ is always a lattice.

**Definition 4 (Meet semilattice)**

*If every pair of elements of a poset $P$ has a meet, we say that $P$ is a meet-semilattice.*

Sometimes, it may be easy to check whether a finite poset is a meet-semilattice. The following proposition then helps us in determining whether the poset is also a lattice.

**Proposition 1**

*Let $P$ be a finite meet semilattice with $\hat{1}$. Then, $P$ is a lattice.*

**Proof.**

We just need to show that given $x, y \in P$, there exists a join of $x$ and $y$.
Towards this end, define $S := \{z \in P : x \leq z,\ y \leq z\}$.
Then, $S$ is finite as $P$ is finite. Moreover, $S$ is nonempty as $\hat{1} \in S$.
Then, it can be seen that $x \vee y = \bigwedge_{z \in S} z$. □

The proof breaks for infinite posets as $S$ defined earlier need not be finite and hence, its meet may not exist.

Analogously, one may define a join-semilattice and the corresponding proposition for a join-semilattice holds as well.

**Definition 5**

*If every subset of $L$ does have a meet and a join, then $L$ is a called a complete lattice.*

(The meet and join of a subset of a lattice have their natural meanings.)
Clearly, a complete lattice has a $\hat{0}$ and $\hat{1}$.

# Modular Lattice

## Proposition 2

*Let $L$ be a finite lattice. The following are equivalent:*

1. *$L$ is graded and the rank generating function $\rho$ of $L$ satisfies $\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$.*
2. *If $x$ and $y$ cover $x \wedge y$, then $x \vee y$ covers $x$ and $y$.*

We omit the proof.
A finite lattice satisfying either of the above (equivalent) properties is called a finite (upper) semimodular lattice.
A finite lattice $L$ whose dual is semimodular is said to be lower semimodular.
A lattice which is both semimodular and lower semimodular is said to be modular.
Thus, a finite lattice is modular if and only if $\rho(x) + \rho(y) = \rho(x \wedge y) + \rho(x \vee y)$ for all $x, y \in L$.

# Distributive lattices

This is the most important class of lattices from a combinatorial point of view.

**Definition 6 (Distributive lattices)**

*A lattice L is said to be distributive if the following laws hold for all $x, y \in L$:*

1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *and*
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

**Examples.**

$[n]$, $B_n$, $D_n$ are distributive lattices.

$\Pi_n$ is not distributive for $n > 2$.

# Order ideals

Recall that the set of all order ideals of a poset $P$, denoted by $J(P)$, ordered by inclusion, forms a poset.
Moreover, $J(P)$ is closed under unions and intersections.
Thus, one can check that $J(P)$ is a lattice as well with $\wedge$ being $\cap$ and $\vee$ being $\cup$.
Now, set theory tells us that $J(P)$ is in fact, a distributive lattice as well.

The Fundamental Theorem of Finite Distributive Lattices (FTFDL) states that the converse is true when $P$ is finite.

**Theorem 1 (FTFDL)**

*Let $L$ be a finite distributive lattice. Then, there is a unique (up to isomorphism) finite poset $P$ for which $L \cong J(P)$.*

The above theorem is also known as *Birkoff's Theorem*.

To prove this theorem, we first produce a candidate $P$ and show that is indeed the case that $J(P) \cong L$. Towards this end, we define the following.

**Definition 7 (Join-irreducible)**

*An element $x$ of a lattice $L$ is said to be join-irreducible if one cannot write $x = y \vee z$ with $y < x$ and $z < x$.*

Equivalently, the above condition says that if $x$ is join-irreducible, then $x = y \vee z$ forces $x = y$ or $x = z$.

# Relation between order ideals and antichains

Before carrying forward, we emphasise the following result from before as it will used often.

## Proposition 3

*Given an order ideal $I$ of a finite poset $P$, there exists a corresponding antichain $A = \{x_1, \dots, x_n\}$ where each $x_i$ is a maximal element of $I$.*
*We also write $I = \langle x_1, \dots, x_n \rangle$.*
*Moreover, $I$ is the smallest order ideal containing $A$.*
*It can also be verified that $\langle x_1, x_2 \rangle = \langle x_1 \rangle \cup \langle x_2 \rangle$.*
*In fact, one has $\langle x_1, \dots, x_n \rangle = \langle x_1 \rangle \cup \dots \cup \langle x_n \rangle$.*

# Link between J(P) and P

The following theorem will help us in coming up with a suitable candidate $P$ and it will also help in showing the uniqueness of $P$ claimed in FTFDL.

**Theorem 2**

*An order ideal $I$ of the finite poset $P$ is join-irreducible* **in $\mathbf{J(P)}$** *if and only if it is a principal order ideal of $P$.*

Before giving a proof of this theorem, we observe that there's a natural one-to-one correspondence between principal order ideals of $P$ and $P$. Namely, $\langle x\rangle \leftrightarrow x$. In fact, this correspondence is also an isomorphism as $\langle x\rangle \subset \langle y\rangle \iff x \leq y$.

Thus, if $J(P) \cong J(Q)$, then the set of join-irreducibles will also be isomorphic and in turn, $P \cong Q$. This shows us that the $P$ mentioned in FTFDL, if it exists, is indeed unique.

# Proof of Theorem 2

**Proof.**

( $\implies$ ) Suppose $I$ is join-irreducible. Since $P$ is finite, Proposition 3 tells us that there exists a corresponding antichain $A$ that generates $I$.
Let us assume that $|A| > 1$. Choose $a \in A$ and let $B := \{a\}$. Then, $\langle A \setminus B\rangle \cup \langle B\rangle = I$.
By Proposition 3, $\langle A \setminus B\rangle \subsetneq I$ and $\langle B\rangle \subsetneq I$.
However, this contradicts that $I$ is join-irreducible. Thus, $|A| = 1$ and hence, $I$ is principal.

( $\impliedby$ ) Suppose $I$ is a principal order ideal. Then, there exists some $x \in P$ such that $\langle x\rangle = P$. Suppose $I = J \cup K$ for some $J, K \in J(P)$.
Then $x \in J$ or $x \in K$. WLOG, we assume that $x \in J$. As $J$ is an order ideal, we get that $\langle x\rangle \subset J$. But $\langle x\rangle = I$. Hence, we get that $J = I$. This proves that $J$ is join-irreducible. □

# Conclusion

What the theorem helps us with is the following -
Suppose that we are given an arbitrary (finite) poset $Q$ and are told that $Q \cong J(P)$ for some poset $P$. The theorem has then shown that the poset $P$ must be isomorphic to the set of the join-irreducibles of $Q$. (Or rather, the subposet obtained by inducing the structure of $Q$ on the set of join-irreducibles of $Q$.)

In effect, it has given us a way of procuring an eligible candidate $P$ to prove the Fundamental theorem that we wanted to prove.

# A helpful lemma

Before proving the theorem, we shall prove the following lemma. Let $L$ be a finite distributive lattice and let $P$ be the set of all the join-irreducible elements of $L$.

**Lemma 1**

*For $y \in L$, there exist $y_1, y_2, \ldots, y_n \in P$ such that $y = y_1 \vee y_2 \vee \cdots \vee y_n$. For $n$ minimal, the expression is unique up to permutations.*

**Proof (Of existence).**

If $y$ is join-irreducible, then we are done.
Suppose $y \notin P$. Then, by definition, there exist $y_1, y_2 \in L$ such that $y = y_1 \vee y_2$ with $y_1 < y$ and $y_2 < y$. If one of $y_1$ or $y_2$ is not in $P$, then we can further "decompose" it. As $L$ is finite and we keep getting smaller elements, this process must stop after a finite number of steps. Thus, given any $y \in L$, there does exist *a* representation as stated. □

# Proof of the lemma

## Proof (of uniqueness).

Now, suppose $y_1 \vee y_2 \vee \cdots \vee y_n = y = z_1 \vee \cdots \vee z_n$ for $y_i, z_i \in P$ for each $i \in [n]$ where $n$ is the least number of elements of $P$ required to be "joined" to get $y$.
Note that given any $i \in [n]$, $z_i \leq y$.

Thus, $z_i = z_i \wedge y = z_i \wedge \left( \bigvee_{j=1}^{n} y_j \right) = \bigvee_{j=1}^{n} (z_i \wedge y_j)$, by distributivity.

But $z_i \in P$ and thus, we get that $z_i = z_i \wedge y_j$ for some $j \in [n]$. This gives us that $z_i \leq y_j$.
Now, suppose it is the case that there exists $k \in [n]$ such that $k \neq i$ and $z_k \leq y_j$. We show that this leads to a contradiction. Since $\vee$ is associative and commutative, we can assume that $i = 1$ and $k = 2$. As $y_j \leq y$, we get that $y_j \vee z_3 \vee \cdots z_n = y$, contradicting the minimality of $n$.
Thus, given any $i \in [n]$, there exists a unique $j \in [n]$ such that $z_i \leq y_j$. Similarly, we get an inequality in the other direction which proves the lemma. □

# Proof of FTFDL

Carrying on with the same notation, we define the following functions:

$$f : J(P) \to L$$
$$f(I) := \bigvee_{x \in I} x.$$

$$g : L \to J(P)$$
$$g(y) := \bigcup_{i=1}^{n} \langle y_i \rangle,$$

where $y = y_1 \vee \cdots \vee y_n$ in the unique way as described earlier. By our previous lemma and commutativity of union, we get that $g$ is indeed well defined.
By our previous lemma, it is also clear that $f$ is surjective.

# Proof of FTFDL

Now, we claim that $g(f(I)) = I$ for every $I \in J(P)$.
To see this, let $A$ be the antichain corresponding to $I$. That is, let $A = \{y_1, \ldots, y_n\}$ where each $y_i$ is a maximal element of $I$.
Then, we get that $f(I) = \bigvee_{x \in I} x = y_1 \vee \cdots \vee y_n$, using the fact that each $a \vee b = b$ if $a \leq b$.
Now, $g(f(I)) = g(y_1 \vee \cdots \vee y_n) = \langle y_1 \rangle \cup \cdots \cup \langle y_n \rangle = \langle y_1, \ldots, y_n \rangle = I. \quad (*)$

Thus, $g \circ f$ is the identity function on $J(P)$ and hence, $f$ is injective.
This shows that $f$ is bijective. As $g$ is its one-sided inverse, it is also its two-sided inverse since $f$ is a bijection. Now, we show that $f$ is an isomorphism by showing that both $f$ and $g$ are order preserving.

# Proof of FTFDL

1. $f$ is order preserving.
Suppose $I_1, I_2 \in J(P)$ with $I_1 \subsetneq I_2$. Then,

$$\begin{aligned} f(I_2) &= \bigvee_{x \in I_2} x \\ &= \left( \bigvee_{x \in I_1} x \right) \vee \left( \bigvee_{x \in I_2 \setminus I_1} x \right) \\ &\geq \bigvee_{x \in I_1} x \\ &= f(I_1) \end{aligned}$$

As we already have seen that $f$ is injective, we get that $f(I_1) < f(I_2)$, as desired.

# Proof of FTFDL

2. $g$ is order preserving.

Suppose that for $I_1, I_2 \in J(P)$, we have $\bigvee_{x \in I_1} x \leq \bigvee_{x \in I_2} x$. We want to show that $I_1 \subset I_2$.

Let $x \in I_1$ be given. We have that $x \leq \bigvee_{x \in I_1} x$ and thus, $x \leq \bigvee_{y \in I_2} y$.

$$\implies x = \left( \bigvee_{y \in I_2} y \right) \wedge x$$

$$= \bigvee_{y \in I_2} (y \wedge x) \qquad \text{(By distributivity)}$$

As $x$ is join-irreducible, there exists some $y \in I_2$ such that $x = y \wedge x$, that is, $x \leq y$. As $I_2$ is an order ideal, this implies that $x \in I_2$.

Thus, $I_1 \subset I_2$. ■

# Small note

The line marked $(*)$ has a possible flaw. We are assuming that $g(y_1 \vee \cdots \vee y_n) = \langle y_1 \rangle \cup \cdots \cup \langle y_n \rangle$, that is, we are assuming that $y_1 \vee \cdots \vee y_n$ is indeed the minimal representation of $f(I)$.
However, this is justified for if there were a shorter representation in terms of join-irreducibles, we would get a contradiction about the maximality of $y_i$s.